Number Systems

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- Different Types of Number Systems
- Conversion Among Number Systems
- Arithmetic Operations
- Decimal Codes
- Alpha-Numeric Codes
- IEEE-754 Floating Point Representation

Data Representation

- There are basically three types of data:
  - **Integer**
  - **Floating-Point**
  - **Character**
- The Integer type is simple.
- The Floating-Point type is bit complicated (IEEE 754 representation)
- The characters are generally encoded (e.g. ASCII encoded)

Types of Number Systems

- Decimal Number System
- Binary Number System
- Octal Number System
- Hexadecimal Number System

Number Systems : General

- A number in base r contains r digits 0, 1, 2, …, r-1, expressed as a power series in r has a general form:

$$A_{n-1}r^{n-1} + A_{n-2}r^{n-2} + \ldots A_1 r^1 + A_0 r^0 + A_{-1}r^{-1} + A_{-2}r^{-2} + \ldots A_{-m+1}r^{-m+1} + A_{-m}r^{-m}$$

- Expressed in positional notation:

$$A_{n-1}A_{n-2}\ldots\ldots A_1A_0 \,.\, A_{-1}A_{-2}\ldots A_{-m+1}A_{-m}$$

- The "." is called the radix point.
- A_{n-1} is referred to as the Most Significant Digit (MSD).
- A_{-m} is referred to as the Least Significant Digit (LSD).
- Uses r distinct digits, hence said to be of base or radix r.
- When m = 0, the LSD $A_{-0} = A_0$.
- To distinguish between number systems enclose the coefficients in parentheses and place subscript after right parenthesis to indicate the base of the number, for example $(23)_{10}$ is a decimal number.

Number Systems : Decimal

- Used in everyday arithmetic to represent numbers by strings of digits.
- Uses 10 distinct digits, hence said to be of base or radix 10.
- A decimal number (of base or radix 10) with n digits to the left and m to the right of the decimal point is represented as:

  $A_{n-1}A_{n-2}.....A_1A_0 . A_{-1}A_{-2}...A_{-m+1}A_{-m}$

  Each A_i is one of {0, 1,2…,9}, i gives the position of the coefficient, hence, the weight 10^i by which the coefficient must be multiplied.
- Example: $(123.4)_{10} = 1 \times 10^2 + 2 \times 10^1 + 3 \times 10^0 + 4 \times 10^{-1}$
- kilo or K – 10^3, mega or M – 10^6, and Giga or G – 10^9

Number Systems : Binary

- Binary number system is a base-2 system with two digits : 0 and 1.
- Most popular representation in digital systems.
- The digits are called bits and the radix point is called the binary point.
- kilo or K – 2^{10}, mega or M – 2^{20}, and Giga or G – 2^{30}
- There are three ways of representing binary numbers :
  - Sign-and-Magnitude
  - 1's Complement
  - 2's Complement

Number Systems : Binary .....

- Both positive and negative numbers can be represented using any of the above three ways.
- In all three methods positive number has identical representations.
- The negative values have different representations.
- In all three ways, if the left most bit is "0" , then the number is positive, whereas if the left most bit is "1" , then number is negative.

Sign-and-Magnitude Binary System

It is simply an ordinary binary number with one extra digit placed in front to represent the sign. If this extra digit is a "1" , it means that the rest of the digits represent a negative number. However, if the same set of digits are used but the extra digit is a "0" , it means that the number is a positive one.

Example: Let us assume that we have an 8-bit register. This means that we have one bit to represent the sign of the number (the **Sign Bit**) and rest 7-bits to represent a number. The numbers are represented as below.

0	0	1	0	0	1	0	1

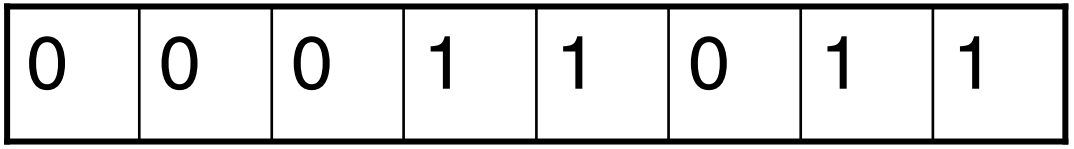
The left most "0" bit means that the number is positive. The rest of the digits represent $(37)_{10}$. Thus, the number represented is $(+37)_{10}$.

1	0	1	0	0	1	0	1

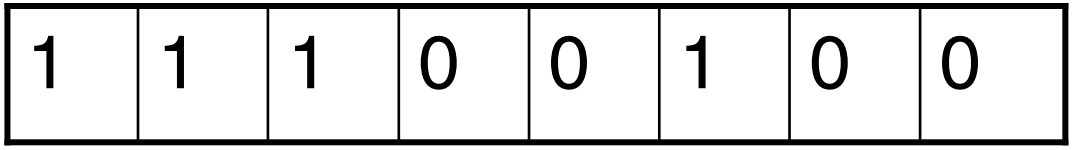
The left most "1" bit means that the number is negative. The rest of the digits represent $(37)_{10}$. Thus, the number represented is $(-37)_{10}$.

Binary System : 1's Complement

The negative values are obtained by complementing each bit of the corresponding positive number.



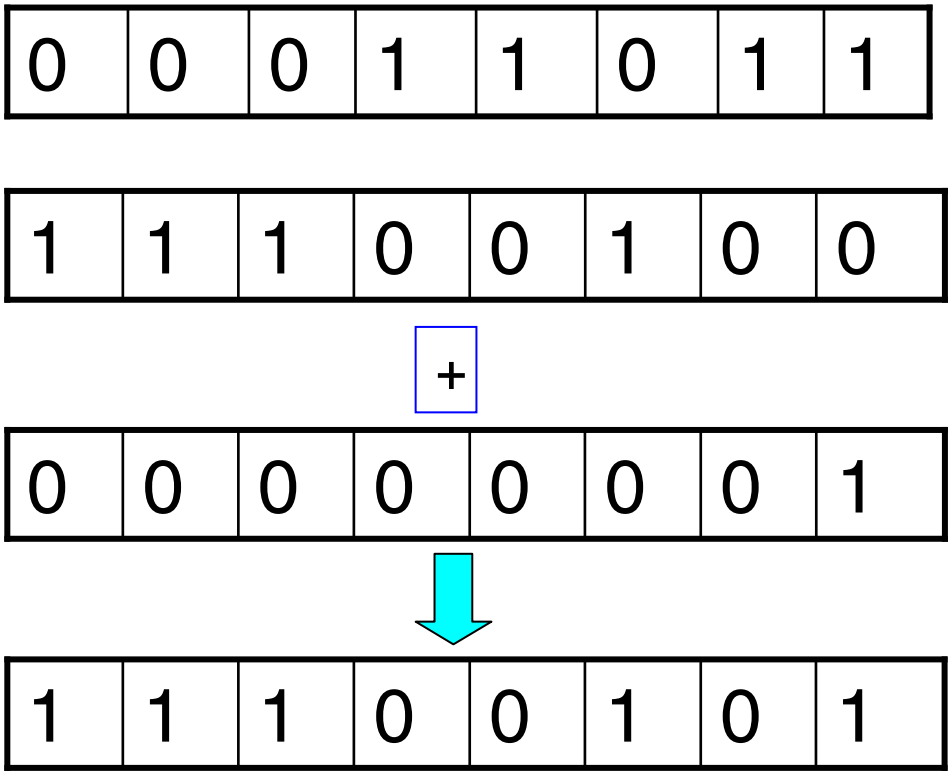
The number represented is $(+27)_{10}$.



The number represented is $(-27)_{10}$.

Binary System : 2's Complement

2's complement of a number is obtained by adding "1" to its 1's complement of the number.



The number represented is $(+27)_{10}$.

1's complement

Add "1" to the above

2's complement number represented is $(-27)_{10}$.

Binary System : 2's Complement ……..

- Every computer system uses 2's complement binary representation. Subtraction can be performed using adders, since (a-b) = a + (-b) = a + ($b^{'}$+1) = a + $b^{'}$+1, thus reducing complexity of hardware implementation.
- Let us assume a 32-bit number : $(a_{31}a_{30}a_{29}a_{28}$ ……..$a_2a_1a_0)_2$. The decimal value of this number is (a_{31} x $(-2)^{31}$ + a_{30} x 2^{30} + a_{29} x 2^{29} + ……. + a_1 x 2^1 + a_0 x 2^0)$_{10}$.
- Example : Decimal value of the 32-bit 2's complement number shown below is $(-4)_{10}$.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0

Number Systems : Octal

- Used for compact representation of binary numbers.
- More convenient to people than using a bit strings in binary.
- The octal number system has radix or base of 8 and uses digits 0, 1, 2, 3, 4, 5, 6, 7.
- Example : $(127.4)_8 = 1 \times 8^2 + 2 \times 8^1 + 7 \times 8^0 + 4 \times 8^{-1} = (87.5)_{10}$.
- Note : The digits 8 and 9 are forbidden.

Number Systems : Hexadecimal

- More compact representation of binary numbers.
- More convenient to people than using a bit strings in binary.
- The hexadecimal number system has radix or base of 16 and uses digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15.
- The letters A, B, C, D, E, and F are used for 10, 11, 12, 13, 14, and 15, respectively.
- Example : $(B65F)_{16} = 11 \times 16^3 + 6 \times 16^2 + 5 \times 16^1 + 15 \times 16^0 = (46687)_{10}$.

First 16 numbers of Decimal, Binary, Octal and Hexadecimal

Decimal (base 10)	Binary (base 2)	Octal (base 8)	Hexadecimal (base 16)
00	0000	00	0
01	0001	01	1
02	0010	02	2
03	0011	03	3
04	0100	04	4
05	0101	05	5
06	0110	06	6
07	0111	07	7
08	1000	10	8
09	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Ranges of Different Numbers

- The range of numbers that can be represented is based on the number of bits available in the hardware structures that store and process information.
- For a computer processing 16-bit unsigned integers, the range of integers is from 0 to $2^{16} - 1$, i.e. from 0 to 65,535.
- For the above computer the range of fractions that can be represented is from 0 to $(2^{16} - 1)/2^{16}$, i.e. from 0.0 to 0.999984712.
- For a computer using 32-bit 2's complement representation the number $(-2,147,483,648)_{10}$, has no corresponding positive number.

Conversion Among Different Number Systems

- Sign Extension of a Binary Number
- Binary to Octal and vice versa
- Binary to Hexadecimal and vice versa
- Decimal Integer to Binary
- Decimal Integer to Octal
- Decimal fraction to Binary
- Decimal fraction to Octal

Conversion : Sign Extension

• Sign extension is conversion of a binary number in *n* bits to a number with *more than n* bits.

•How to do ? Take the MSB from the smaller quantity and replicate it to the new bits of the larger quantity, and then copy the old bits of the smaller quantity into the right portion of the new word.

•Example :

4 bits Number	8-bit Number
$(0010)_2$	$(0000\ 0010)_2$
$(1010)_2$	$(1111\ 1010)_2$

Conversion : Binary ⟺ Octal

- Partition the binary number into groups of three each starting from the binary point and proceed to the left and to the right.
- Assign octal digit to each group.
- Example:

$(010110001101011.111100000110)_2$ (Given Number)

$(010\ 110\ 001\ 101\ 011\ .\ 111\ 100\ 000\ 110)_2$ (Grouped into 3)

$(2\ \ 6\ \ 1\ \ 5\ \ 3\ \ .\ \ 7\ \ 4\ \ 0\ \ 6)_8$ (Octal Number)

- The above process can be reversed to convert from the octal numbers to binary numbers.

Conversion : Binary ⟺ Hexadecimal

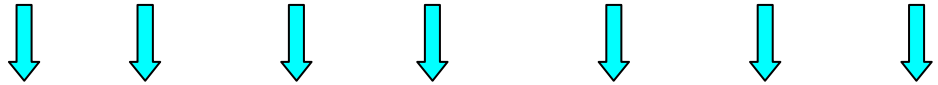
•Partition the binary number into groups of four each starting from the binary point and proceed to the left and to the right.

•Assign hexadecimal digit to each group.

•Example:

$(0010110001101011 . 1111000001 10)_2$ (Binary Number)

$(0010\ 1100\ 0110\ 1011 . 1111\ 0000\ 0110)_2$ (Grouped into 4)



$(2\quad C\quad 6\quad B\quad .\quad F\quad 0\quad 6)_{16}$ (Hexadecimal Digits)

$(2C6B.F06)_{16}$ (Hexadecimal Number)

•The above process can be reversed to convert from the hexadecimal numbers to binary numbers.

Some Arithmetic Terms

	Addition	Subtraction	Multiplication	Division
Numerator	Augend	Minuend	Multiplicand	Dividend
Denominator	Addend	Subtrahend	Multiplier	Divisor
Result	Sum	Difference	Product	Quotient

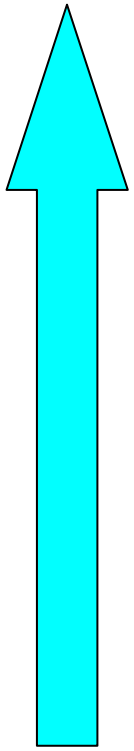
Conversion : Some General Rules

- The conversion of a number in base *r* to a decimal is done by expanding the number in a power series of *r* and adding all the terms.
- The reverse process, i.e. conversion of decimal integer to any base *r* is bit complicated.
- In the number contains a decimal point, then the number has to be divided into integer part and fraction part, and the conversion of the two parts should be done separately.
- The conversion of decimal integer to base *r* is done by dividing the number and the successive coefficients by *r*, and accumulating the remainders.
- The conversion of the decimal fraction to base *r* is done by multiplying the number and the fraction part of the successive products, and accumulating the integer part of the successive products.

Conversion : Decimal Integer to Binary

Let us convert $(1697)_{10}$ to base 2 :

1697	/ 2 =	848	remainder 1
848	/ 2 =	424	remainder 0
424	/ 2 =	212	remainder 0
212	/ 2 =	106	remainder 0
106	/ 2 =	53	remainder 0
53	/ 2 =	26	remainder 1
26	/ 2 =	13	remainder 0
13	/ 2 =	6	remainder 1
6	/ 2 =	3	remainder 0
3	/ 2 =	1	remainder 1
1	/ 2 =	0	remainder 1



(Least Significant Digit)

(Most Significant Digit)

$(1697)_{10} = (11010100001)_2$

Note : The same procedure can be used to convert an decimal integer to base 16 or 8, by dividing the decimal integer by 16 or 8, instead of 2.

Conversion : Decimal Integer to Octal

Let us convert $(44978)_{10}$ to base 8 :

Division	Quotient	Remainder	Octal Number
44978 / 8	5622	2	2
5622 / 8	702	6	62
702 / 8	87	6	662
87 / 8	10	7	7662
10 / 8	1	2	27662
1 / 8	0	1	127662

$$(44978)_{10} = (127662)_8$$

Note : The same procedure can be used to convert an decimal integer to base 16, by dividing the decimal integer by 16, instead of 8.

Conversion : Decimal Integer to Hexadecimal

Let us convert $(44978)_{10}$ to base 16 :

Division	Quotient	Remainder	Hexadecimal Number
44978 / 16	2811	2	2
2811 / 16	175	11	B2
175 / 16	10	15	FB2
10 / 16	0	10	0AFB2

$$(44978)_{10} = (0AFB2)_{16}$$

Conversion : Decimal Fraction to Binary

Let us convert $(0.375)_{10}$ to binary fraction. Following are the steps to convert this number to binary using repeated multiplication by 2.

Step1 : Multiply 0.375 by 2 to find the MSB. Since our result is less than 1, the MSB in our answer is 0. 0.375*2 = 0.75 Answer: 0 . 0 ? ?

Step2: Take the fractional part of the previous result (0.75) and multiply by 2 again. Now the result is greater than 1, so the next bit of is 1. 0.75*2 = 1.5 Answer: 0 . 0 1 ?

Step3: Take the fractional part of the previous result (0.5) and multiply by 2. This time the result is exactly 1, so the LSB is 1. Since the fractional part of our result is 0, this is the last multiplication needed to find our answer. 0.5*2 = 1.0 Answer: 0 . 0 1 1

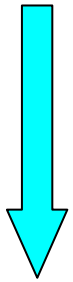
$$(0.375)_{10} = (0.011)_2$$

Conversion : Decimal Fraction to Octal

To convert a decimal fraction to octal, *multiply* the fraction by 8. Extract everything that appears to the left of the radix point. The first number extracted will be the MSD and will follow the radix point. The last number extracted will be the LSD.

Example: Convert decimal $(0.513)_{10}$ to a three digit octal fraction.

0.513 x 8 = 4.104	Integer = 4	(MSD)
0.104 x 8 = 0.832	= 0	
0.832 x 8 = 6.656	= 6	
0.656 x 8 = 5.248	= 5	(LSD)



$$(0.513)_{10} = (0.4065)_8 = (0.407)_8$$

Conversion : Decimal Fraction to Hexadecimal

To convert a decimal fraction to hexadecimal, *multiply* the fraction by 16. Extract everything that appears to the left of the radix point. The first number extracted will be the MSD and will the last number extracted will be the LSD.

Example : Convert decimal $(0.513)_{10}$ to a three digit hexadecimal fraction.

0.513 x 16 = 8.208	Integer = 8	(MSD)
0.208 x 16 = 3.328	= 3	
0.328 x 16 = 0.052	= 0	
0.052 x 16 = 0.832	= 0	(LSD)

$$(0.513)_{10} = (0.830)_{16}$$

Arithmetic Operations

- Binary addition
- Binary subtraction
- Binary multiplication
- Hexadecimal addition
- Octal multiplication

Arithmetic : Binary addition

- The sum of two numbers is calculated following the same rules as for decimal numbers.
- Unlike the decimal system the sum of two bits has to be only 1 or 0 (in decimal : 0…..to 9).
- The carry occurs if the sum is greater than 1 (decimal carry occurs if sum is greater than 9).
- Example :

Carries :	00000	101100
Augend :	01100	10110
Addend :	+10001	+10111
Sum :	11101	101101

- The second addition has a carry.

Arithmetic Operations : Binary subtraction

- The rules of binary subtraction are the same as that of decimal numbers.
- A borrow into a given column adds 2 to the minuend bit (a borrow in the decimal system adds 10 to the minuend digit.
- Example:

Borrows :	00000	00110
Minuend :	10110	10110
Subtrahend:	-10010	-10011
Difference :	11101	00011

- Subtraction can be also performed by adding the 2's complement of the subtrahend to the minuend.

Arithmetic Operations : Binary multiplication

- Example:

  Multiplicand: 1011

  Multiplier: x 101

  1011

  0000

  1011

  Product: 110111

- Observation : The multiplier bits are always 1 or 0, therefore the partial products are equal to either the multiplicand or to 0.
- The above fact has been exploited in various ways, and many time and hardware efficient multiplication algorithms have been developed.
- Booth's multiplier and Wallace-Tree multiplier are two examples.

Arithmetic Operations : Hexadecimal addition

Example : Perform the addition $(59F)_{16} + (E46)_{16}$

Hexadecimal Addition		Equivalent Decimal Calculation	
		1	
59F	5	9	15 (F)
E46	14 (E)	4	6
13E5	19=16+3	14=E	21=16+5

The equivalent decimal calculations columns shown in the right hand side gives mental reasoning that must be carried out to produce each digit of the hexadecimal sum.

Arithmetic Operations : Octal multiplication

Perform the multiplication $(762)_8$ x $(45)_8$

Octal	Octal		Decimal		Octal
762	5x2	=	10 = 8+2	=	12
45	5x6+1	=	31 = 24+7	=	37
4672	5x7+3	=	38 = 32+6	=	46
3710	4x2	=	8 = 8+0	=	10
43772	4x6+1	=	25 = 24+1	=	31
	4x7+3	=	31 = 24+7	=	37

The computations on the right show the mental calculations for each pair of the octal digit.

Overflow handling during arithmetic operations

- Overflow occurs when the result too large for finite computer word (i.e. bit-width of registers and computational units).
- Overflow term is somewhat misleading, it does not mean a carry "overflowed".
- No overflow when adding a positive and a negative number and when signs are the same for subtraction.
- Effects of Overflow : An exception (interrupt) occurs
  - Control jumps to predefined address for exception
  - Interrupted address is saved for possible resumption
- Many hardware supports are available with different computer architectures to handle the overflow.

Why Decimal or Alphanumeric Codes ??

Binary number system is used in computers, but people are more comfortable with decimal system. So, there has to be back and forth conversion between decimal and binary system. The computation is done in binary and human understanding is in decimal. This needs a method to store decimals (and characters) in a computer that can be converted to binary.

Decimal Codes

An n-bit binary code is a group of n-bits that assume up to 2^n distinct combinations of 1's and 0's, with each combination representing one element of the set being coded. The most commonly used for decimal is binary-coded-decimal (BCD).

TABLE 1-3
Binary-Coded Decimal (BCD)

Decimal Symbol	BCD Digit
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

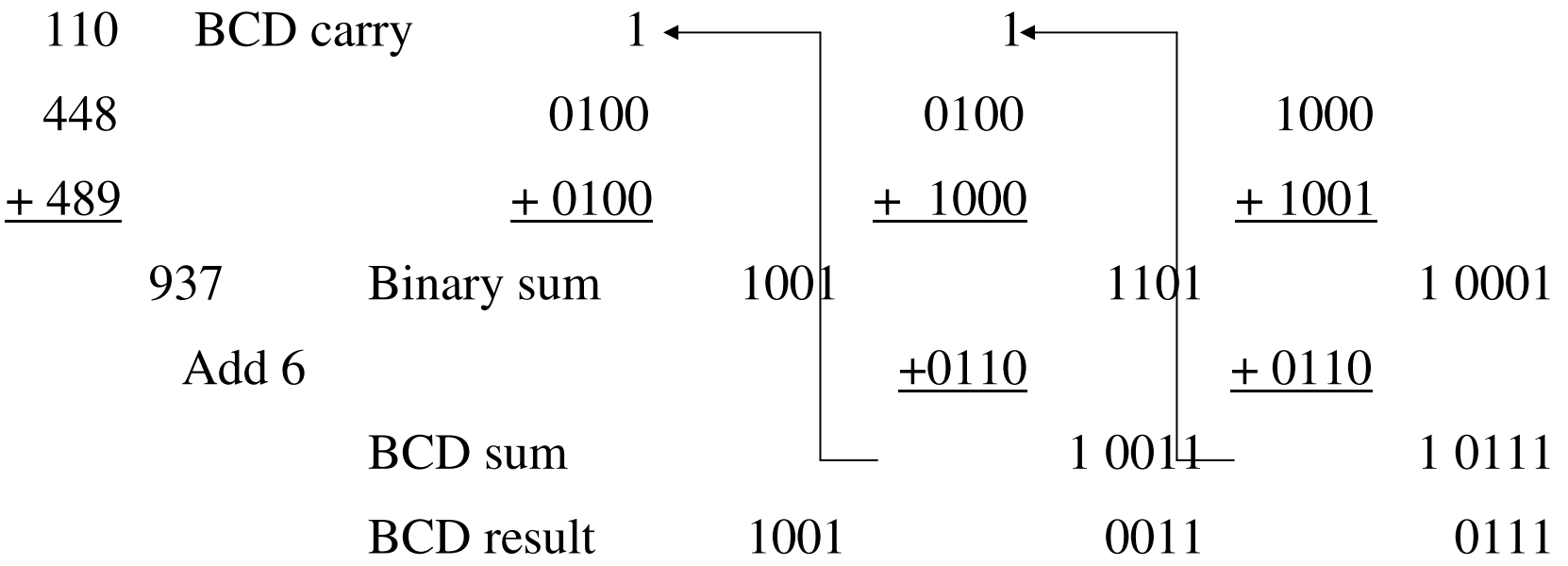
Table 1-3 Binary-Coded Decimal (BCD)

BCD addition

•In each position two BCD digits are added as if they were two binary numbers.

•If the binary sum is greater than 1001, 0110 is added to obtain the correct BCD digit sum and carry.

•Example :



Alphanumeric Codes

TABLE 1-4
American Standard Code for Information Interchange (ASCII)

$B_4B_3B_2B_1$	$B_7B_6B_5$ = 000	001	010	011	100	101	110	111
0000	NULL	DLE	SP	0	@	P	`	p
0001	SOH	DC1	!	1	A	Q	a	q
0010	STX	DC2	"	2	B	R	b	r
0011	ETX	DC3	#	3	C	S	c	s
0100	EOT	DC4	$	4	D	T	d	t
0101	ENQ	NAK	%	5	E	U	e	u
0110	ACK	SYN	&	6	F	V	f	v
0111	BEL	ETB	'	7	G	W	g	w
1000	BS	CAN	(	8	H	X	h	x
1001	HT	EM	)	9	I	Y	i	y
1010	LF	SUB	*	:	J	Z	j	z
1011	VT	ESC	+	;	K	[	k	{
1100	FF	FS	,	<	L	\	l	|
1101	CR	GS	-	=	M	]	m	}
1110	SO	RS	.	>	N	^	n	~
1111	SI	US	/	?	O	_	o	DEL

Control Characters:

NULL	NULL	DLE	Data link escape
SOH	Start of heading	DC1	Device control 1
STX	Start of text	DC2	Device control 2
ETX	End of text	DC3	Device control 3
EOT	End of transmission	DC4	Device control 4
ENQ	Enquiry	NAK	Negative acknowledge
ACK	Acknowledge	SYN	Synchronous idle
BEL	Bell	ETB	End of transmission block
BS	Backspace	CAN	Cancel
HT	Horizontal tab	EM	End of medium
LF	Line feed	SUB	Substitute
VT	Vertical tab	ESC	Escape
FF	Form feed	FS	File separator
CR	Carriage return	GS	Group separator
SO	Shift out	RS	Record separator
SI	Shift in	US	Unit separator
SP	Space	DEL	Delete

Table 1-4 American Standard Code for Information Interchange (ASCII)

ASCII is a 7-bit code, but one bit ("0") is appended as MSB to make it a byte.

Parity Bit

- A parity bit is an extra bit included to make total number of 1's either even or odd.
- An 8th bit is added to the ASCII character to indicate its parity.
- Useful to detect errors in data communications and processing. The parity at the sending and the receiving end must match.
- Example :

		Even Parity	Odd Parity
ASCII A =	1000001	01000001	11000001
ASCII T =	1010100	11010100	01010100

- Use of even parity is more common.

Floating Point Numbers : A brief look

- We need a way to represent
  - numbers with fractions, e.g., 3.1416
  - very small numbers, e.g., .000000001
  - very large numbers, e.g., 3.15576×10^9
- Representation:
  - sign, exponent, significand: $(-1)^{sign}$ x significand x $2^{exponent}$
  - more bits for significand gives more accuracy
  - more bits for exponent increases range
- IEEE 754 floating point standard:
  - single precision: 8 bit exponent, 23 bit significand
  - double precision: 11 bit exponent, 52 bit significand

IEEE 754 floating-point standard

- Leading "1" bit of significand is implicit
- Exponent is "biased" to make sorting easier
  - all 0s is smallest exponent all 1s is largest
  - bias of 127 for single precision and 1023 for double precision
  - summary: $(-1)^{sign}$ x (1+significand) x $2^{exponent - bias}$
- Example:
  - decimal: -.75 = -3/4 = $-3/2^2$
  - binary: -.11 = -1.1×2^{-1}
  - floating point: exponent = 126 = 01111110
  - IEEE single precision: 10111111010000000000000000000000